B.sc(H) part 3 paper 5
Topic:Defination& existence of Rie
man integral of bounded function
Subject mathematics

Def .. Let f be a bounded real function on the closed and bounded interval $[a, b]$. We say that the Riemann interval of f exists or f is Riemann integrable (or R-integrable) or integrable over $[a, b]$ if its lower and upper Riemann integrals are equal

i.e. if $\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx;$

and the common value of these integrals is called the **Riemann integral** (or simply R-integral) of f over $[a, b]$ and is denoted by the symbol $\int_a^b f(x)\, dx.$

For simplicity we sometimes denote it by $\int_a^b f$. The above definition of R-integrability is called the bounded definition.

Remark 1. The symbol R $[a, b]$ shall denoted the class of all real bounded functions f which are Riemann integrable over $[a, b]$. The numbers a and b may be termed as lower and upper limits of integration respectively.

Remark 2. The statement that $\int_a^b f(x)\, dx$ exists shall mean that f is bounded and R—integrable over $[a, b]$.

Remark 3. If $a=b$, we define $\int_a^a f(x)\, dx=0$. If $b<a$ then we define $\int_a^b f dx = -\int_b^a f dx$ whenever f is R-integrable in $[b, a]$.

Theorem If a bounded function f is R-integrable over $[a, b]$ and M, m are the bounds of f then

$$m(b-a) \leqslant \int_a^b f(x)\, dx \leqslant M\,(b-a) \text{ if } b \geqslant a$$

$$\& \; m(b-a) \geqslant \int_a^b f(x)\, dx \geqslant M(b-a), \text{ if } b \leqslant a.$$

Proof. Let $b \geqslant a$. Since f is R-integrable so

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx$$

Hence $lub\,\{L(P)\} = glb\,\{U(P)\} = \int_a^b f(x)dx$ for all possible partitions P of $[a, b]$.

Also by Th. 1·2, $\{L(P)\}$ is bounded above by $M(b-a)$ and $\{U(P)\}$ is bounded below by $m(b-a)$. Hence

$$m(b-a) \leqslant glb\{U(P)\} = lub\,\{L(P)\} \leqslant M(b-a).$$

$$\therefore\ m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a) \text{ if } b \geqslant a.$$

If $b \leqslant a$, then $a \geqslant b$, hence as proved above

$$m(a-b) \leqslant \int_b^a f(x)dx \leqslant M(a-b),$$

$$\text{hence } -m(a-b) \geqslant -\int_b^a f(x)dx \geqslant -M(a-b),$$

$$\therefore\ m(b-a) \geqslant \int_a^b f(x)dx \geqslant M(b-a) \text{ if } b \leqslant a.$$

Corollary 1. If f is bounded and R-integrable in $[a, b]$ then there exists a number λ lying between the bounds of f such that

$$\int_a^b f(x)dx = \lambda(b-a).$$

Remark 3. Note that the concept of Riemann integrability has been introduced under two very important restrictions :

(a) The function should be bounded;

(b) The interval of integration is of finite length so that neither of the end points of the interval is infinite.